# Qualifying exam - 2010

# Statistical Mechanics

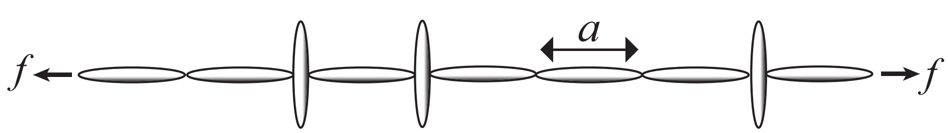
NAME:____________________________

You can use a graduate level textbook, a calculator and a unit conversion table. Please write legibly and show all steps of your derivations and/or calculations.

**Problem 1** [15 points]

Consider a simple model of a polymer chain composed of $N \gg 1$ identical molecules of length $a$ as in the figure below. Each molecule can be aligned either along the chain or normal to it. In the latter case its projection on the chain direction is assumed to be zero. Each molecule can be only in one these two states. Each state is degenerate and has the same energy in the absence of forces. Kinetic energy of the molecules can be neglected. A tension force $f$ is applied parallel to the chain and the system is equilibrated with a thermostat.

1. Find the entropy $S$ and length $L$ of the polymer chain as functions of temperature $T$ and force $f$.
2. In the limit of small force, show that $L$ becomes a linear function of $f$ at a fixed $T$. This relation is similar to Hooke's law of elasticity.




**Problem 2** [25 points]

Consider $N$ fixed non-interacting magnetic moments of magnitude $\mu$. The system is in a uniform external magnetic field B and in equilibrium with a thermostat at a temperature $T$. Assuming that each magnetic moment can be oriented only parallel or anti-parallel to the field, derive analytical expressions for:

1. Specific heat of the system
2. Average magnetic moment $\overline{M}$
3. Fluctuation of the magnetic moment $\Delta M = \sqrt{\overline{\left(M - \overline{M}\right)^2}}$

4. Magnetic susceptibility $\chi = d\overline{M}/dB$

5. Find the asymptotic behaviors of $\chi$ in the limits of high temperatures and low temperatures

**Problem 3** [25 points]

Considering ammonia as an ideal gas whose molecules $NH_3$ have the structure of a triangular pyramid,

1. Derive analytical expressions for:

   (a) the chemical potential of ammonia at a given temperature $T$ and pressure $p$,

   (b) constant-volume specific heat ($C_V$) per molecule of ammonia as a function of $T$.

2. Compute $C_V$ per mole of ammonia (in J/mol/K) at the temperatures of 300 K and 1500 K. Explain the difference between the results.

3. Compute the high-temperature limit of $C_V$ for ammonia (in J/mol/K).

Include only translational, rotational and vibrational degrees of freedom. The rotations can be treated in the classical limit. The $NH_3$ molecule has six normal frequencies of atomic vibrations. For numerical calculations, assume that all normal frequencies are equal to $\nu = 60$ THz.

**Problem 4** [35 points]

1. Consider a gas of free electrons at a temperature much smaller than the Fermi temperature. Present a detailed, step by step derivation of the following relations:

$$u = \frac{3}{5}\varepsilon_F \quad (1)$$

$$p = \frac{2}{3}nu \quad (2)$$

$$C = \frac{\pi^2 k^2 T}{2\varepsilon_F} \quad (3)$$

where $\varepsilon_F$ is the Fermi energy, $u$ is the average energy per electron, $n$ is the number density of electrons, $p$ is pressure of the gas, $C$ is specific heat per electron, and $k$ is Boltzmann's constant.

2. Suppose the gas contains 15 electrons per cubic nanometer (typical electron density in metals). Compute the Fermi energy (in eV) and the specific heat per electron (in eV/K) at a temperature of 300 K. Compare with specific heat of a classical gas per particle. Explain the difference.