Classical Mechanics Qualifier (January 2013)
## George Mason University

You will have **THREE** hours to complete the exam.
You are allowed to use your graduate textbook during the exam.

Short Problems – do both (15 points each)

Problem 1

A thin uniform disc with radius a and mass m is rotating wth a constant angular velocity ω about the fixed vertical axis passing through its center. ω is inclined at a constant angle α with respect to the symmetry axis z' of the disc. What is the magnitude and direction of the torque with respect to the lab frame needed to keep the disc in this motion?
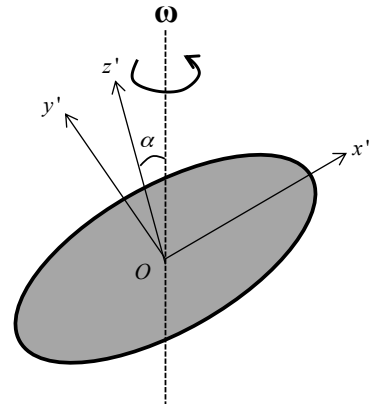
Problem 2

a) Write down the Hamiltonian for a freely falling particle with mass m and gravitational constant g. (Vertical motion only)

b) Using the Poisson bracket, show that the following dynamical variable
$$F = x - \frac{pt}{m} - \frac{gt^2}{2}$$
is a constant of motion. Here, p is the momentum.

Long Problems – do two of the following three (30 points each):

Problem 3

A solid homogeneous cylinder with radius a and mass m rolls without slipping on the inside of a stationary larger cylindrical tube with radius R as shown in the diagram below.

a) Write down the Lagrangian for the problem.
b) Write down the holonomic constraint for the "rolling without slipping" condition.
c) Determine the equation of motion for the rolling cylinder in a single generalized coordinate.
d) Find the frequency of small oscillations about the stable equilibrium at the bottom of the tube.
e) What is the magnitude of the static friction responsible for the rolling of the cylinder without slipping motion?
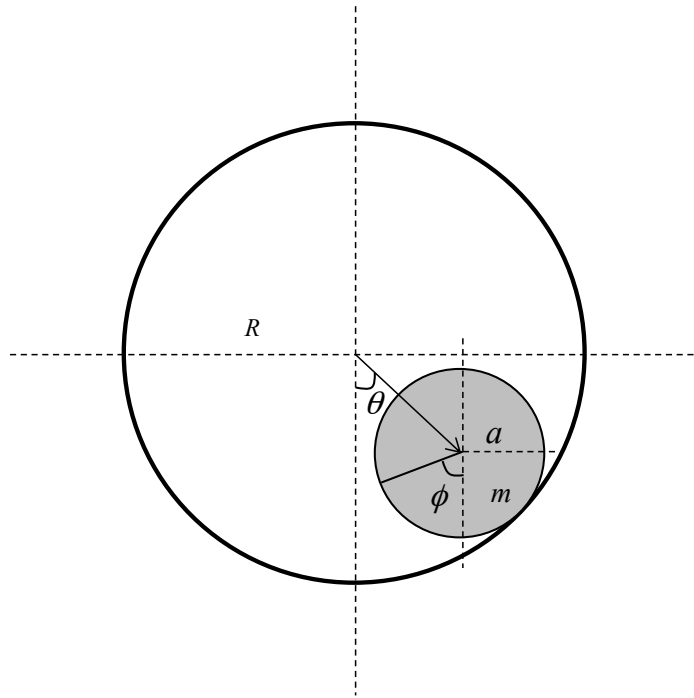
Problem 4

a) A system with a single degree of freedom is described by the following Hamiltonian:
$$H = i\omega qp,$$
where $i = \sqrt{-1}$, ω is a real parameter. Using the Hamilton's equations, solve for the system's trajectory in phase space, $q(t)$ and $p(t)$, with initial conditions, $q(0) = q_0$ and $p(0) = -i\omega q_0/2$.

b) Use the following generating function
$$F = qP + \frac{i}{2}\frac{P^2}{\omega} - \frac{i}{4}\omega q^2$$
to derive a canonical transformation: $Q = Q(q,p)$ and $P = P(q,p)$ for the system in part a). Show that the transformed Hamiltonian $K(Q,P)$ describes a simple harmonic oscillator.

c) You are given another Hamiltonian:
$$\hat{H} = \frac{\omega^2}{2\hat{q}^2} + \frac{\hat{p}^2\hat{q}^4}{2}.$$
Use the Hamilton's equations to write down the equations of motion for $\hat{H}$.

d) Show that the following transformation
$$Q = -\frac{1}{\hat{q}} \quad \text{and} \quad P = \hat{p}\hat{q}^2$$
is canonical and it takes $\hat{H}$ to the same Hamiltonian $K(Q,P)$ for the harmonic oscillator found in part b).

Problem 5

Two masses, $2m$ and m are suspended from a fixed hinge by two massless springs with the same spring constant k as shown on the right. The unstretched lengths of the two springs are l_1 and l_2.

What are the normal modes and the corresponding characteristic frequencies for the system? (Consider vertical motion only.)