Quantum Mechanics, Qualifying Exam, Jan. 2012

Name:

Note: This is an open book exam and you are allowed to bring Sakurai or Shankar's book on Quantum Mechanics. If a formula appears in the book, please use that as a starting point; there is no need to show the derivation of that formula.

(1)

Consider a system described by the Hamiltonian H,

$$H = b \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

where b is a constant with dimension of energy.

(a) At $t = 0$, we measure the energy of the system. What possible values will we obtain? [5 pts]

(b) At later time t, we measure the energy again. How is it related to its value we obtained at $t = 0$? [5 pts]

(c) Suppose at $t = 0$, the system is equally likely to be in its two possible energy eigenstates. Write down the most general state of the system at $t = 0$. Taking this state as the initial state, find the state at $t = 10\,\hbar/b$. What is the probability that the system at $t = 10\,\hbar/b$ is in a state different from its initial state? [10pts]

(2)

At $t = 0$, a particle of mass m confined in a one-dimensional potential well is in an energy eigenstate with wave function $\psi(x) = Ae^{-(x/b+3)^2}$, where b is a constant with dimension of length. You can use the following integrals:

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}, \quad \int_{-\infty}^{+\infty} x^2 e^{-\alpha x^2} dx = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}$$

(a) Determine the normalization constant A. [5 pts]

(b) Where is the particle most likely to be found? [5 pts]

(c) Calculate $\langle x \rangle$, $\langle p \rangle$, $\langle x^2 \rangle$, $\langle p^2 \rangle$, and the uncertainty $\Delta x \Delta p$ in this state.

[10 pts]
(d) If $\psi(x)$ is the ground state wave function, find the Hamiltonian of the particle, and the ground state energy. [10 pts]

(3) Short questions [20 pts, 5 pts each].
Let $\hat{x}$ be the position operator, and $\hat{p}$ be the momentum operator, in one-dimension.
(a) Write down the form of $\hat{x}$ and $\hat{p}$ in the x-basis.
(b) Write down the form of $\hat{x}$ and $\hat{p}$ in the p-basis.
Evaluate the following,
(c) $e^{-i\hat{p}L/\hbar}|x\rangle =$
(d) $[\hat{p}, e^{-ikx}] =$
where L and k are constants.

(4)
A spinless particle in a spherically symmetric potential is described by a wave function, $\psi(x, y, z) = A[1 + (x + z)/r]$ where A is a normalization constant.
(a) Find the possible angular momentum quantum numbers, l and m, of the system. [5 pts]
(b) Calculate the probability of the system being found in each angular momentum eigenstates labeled by l and m. [10 pts]

(5)
A particle moving in three dimensions is subject to a potential $V(r) = V_0 \log(r/a)$, where r is the radial distance from the origin, V_0 and a are constants.
(a) What is the angular part of the wave function $\psi(\theta, \phi)$ for angular momentum $l = 3$, $m = 3$? [5 pts]
(b) Does the eigen energy depend on l and m? Explain your answer using the symmetry of the Hamiltonian. [5pts]
(c) Does the answer to (b) change if we replace $V(r)$ with $V(r) = V_0 a/r$? Explain why. [5pts]