Quantum Mechanics Qualifying Exam

Spring 2017

January 19 (9:00 am - 12:00 pm), Exploratory Hall 1004

1. Consider a particle with a wavefunction:

$$\psi(x,y,z) = N(x+y+z)e^{-(x^2+y^2+z^2)/\alpha^2}$$

where N is a normalization constant and α is a length-scale parameter. We measure the values of L^2 and L_z. Find the probabilities that the measurements yield:

(a) $L^2 \to 2\hbar^2$, $L_z \to 0$.

(b) $L^2 \to 2\hbar^2$, $L_z \to \hbar$.

(c) $L^2 \to 2\hbar^2$, $L_z \to -\hbar$.

Use the formulas for normalized spherical harmonics:

$$Y_1^1(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi} \quad , \quad Y_1^0(\theta,\phi) = -\sqrt{\frac{3}{4\pi}}\cos\theta \quad , \quad Y_1^{-1}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi}$$

2. Suppose we have a system with total angular momentum $l = 1$. The ensemble of states is described by the density matrix:

$$\rho = \frac{1}{4}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

in the representation that diagonalizes L_z.

(a) Is ρ a permissible density matrix. Does it describe a pure or mixed ensemble?

(b) What are the average values of the measured L_x, L_y, L_z in this ensemble?

(c) What is the standard deviation (uncertainty) ΔL_z of the measured L_z?

3. A free particle of mass m, moving through a one-dimensional world, is initially in the state given by the wavefunction:

$$\psi(x,0) = A\exp\left(-\frac{x^2}{a^2}\right)$$

(a) Find the probability amplitude in momentum space $\psi(p,0)$ at $t=0$.

(b) Find $\psi(x,t)$ at all times t.

Don't bother calculating the normalization constant A, but use the assumption that $\psi(x,t)$ is normalized to eliminate A from the final result. You may use the formula:

$$\int_{-\infty}^{\infty} dx\, e^{-\lambda x^2} = \sqrt{\frac{\pi}{\lambda}}$$

4. The Hamiltonian for a harmonic oscillator can be written in dimensionless units ($m = \hbar = \omega = 1$) as:

$$H = a^\dagger a + \frac{1}{2}$$

where:

$$a = \frac{x + ip}{\sqrt{2}} \quad , \quad a^\dagger = \frac{x - ip}{\sqrt{2}}$$

One unnormalized energy eigenfunction is:

$$\psi = (2x^3 - 3x)e^{-x^2/2}$$

Derive two other (unnormalized) eigenfunctions which are closest in energy to ψ. Stating solutions without proof will earn no credit.