Quantum Mechanics Qualifying Exam

Fall 2017

August 24 (9:00 am - 12:00 pm), Exploratory Hall 1004

1. (a) A particle of mass m is subjected to a force $\mathbf{F}(\mathbf{r})$ in three dimensions such that the wavefunction $\psi(\mathbf{p}) = \langle \mathbf{p}|\psi\rangle$ in momentum representation satisfies the momentum-space Schrodinger equation:

$$\left(\frac{p^2}{2m} - a\nabla^2_{\mathbf{p}}\right)\psi(\mathbf{p}) = i\hbar\frac{\partial\psi(\mathbf{p})}{\partial t}$$

where the momentum-space gradient at momentum $\mathbf{p} = (p_x, p_y, p_z)$ is:

$$\boldsymbol{\nabla}_{\mathbf{p}} = \hat{\mathbf{x}}\frac{\partial}{\partial p_x} + \hat{\mathbf{y}}\frac{\partial}{\partial p_y} + \hat{\mathbf{z}}\frac{\partial}{\partial p_z}$$

Find the force $F(\mathbf{r})$ in real space.

2. A particle of mass m free to move in only one dimension is perfectly confined to a region $0 < x < a$ where no force acts on it. At $t = 0$ its normalized wavefunction is:

$$\psi(x, 0) = \sqrt{\frac{8}{5a}}\left[1 + \cos\left(\frac{\pi x}{a}\right)\right]\sin\left(\frac{\pi x}{a}\right)$$

What is the wavefunction $\psi(x, t)$ at any later time $t > 0$? What is the average energy of the particle at an arbitrary time $t > 0$?

3. For electronic states in a finite one-dimensional system of size N, a simple model Hamiltonian is:

$$H = \sum_{n=1}^{N}\left[E_0|n\rangle\langle n| + W\Big(|n\rangle\langle n+1| + h.c.\Big)\right]$$

where the states $|n\rangle$ make an orthonormal basis (i.e. $\langle n|n'\rangle = \delta_{nn'}$) and E_0, W are parameters. Assume periodic boundary conditions $|n+N\rangle \equiv |n\rangle$. Calculate the energy levels and wavefunctions of all stationary states.

4. A particle of mass m is constrained to move between two concentric impermeable spheres of radii $r = a$ and $r = b$. There is no other potential. Find the ground-state energy and normalized wavefunction.

5. Two electrons are tightly bound to different neighboring sites in a certain solid. They are, therefore, distinguishable particles which can be described in terms of their respective Pauli spin matrices $\boldsymbol{\sigma}^{(1)}$ and $\boldsymbol{\sigma}^{(2)}$. The Hamiltonian of these two electrons is:

$$H = -J\left[\sigma_x^{(1)}\sigma_x^{(2)} + \sigma_y^{(1)}\sigma_y^{(2)}\right]$$

where J is a constant.

(a) Find the spectrum and degeneracy of all energy levels.

(b) Introduce a magnetic field of magnitude B in the z direction. How are the energy levels modified? Draw an energy level diagram as a function of B.