

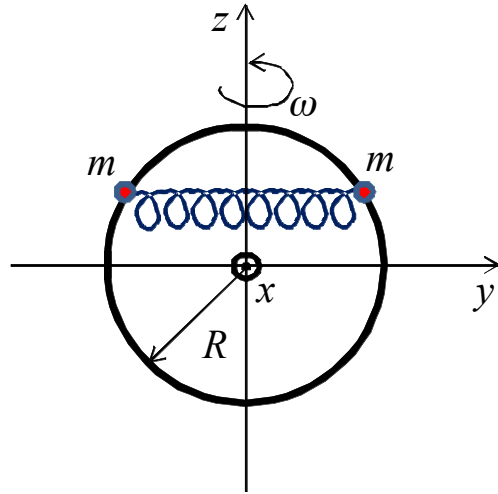
# Classical Mechanics Qualifier (August 2011)

## George Mason University

You will have **THREE** hours to complete the exam.  
You are allowed to use your graduate textbook during the exam.

### Problem 1 (30pts)

Two beads with equal mass  $m$  connected by a spring are restricted to slide on a circular hoop with radius  $R$  as shown. The spring has an equilibrium length of  $2r_0$  and a force constant  $k$ .  $r_0$  is assumed to be less than  $R$ . The hoop rotates about the  $z$ -axis with a constant angular velocity  $\omega$ . For simplicity, we assume that the spring will remain parallel to the  $xy$ -plane as the hoop spins. (This is a reasonable assumption for two equal masses starting with the same  $z$  coordinates.) There is no gravity.



- a) (8 pts) Using cylindrical coordinates  $(r, \phi, z)$ , show that the Lagrangian can be written in the following one-dimensional form in terms of a single generalized coordinate  $z$ :

$$L = \frac{1}{2} \mu(z) \dot{z}^2 - V_{\text{eff}}(z)$$

where

$$\mu(z) = 2m(1 - z^2/R^2)^{-1} \quad \text{and}$$

$$V_{\text{eff}}(z) = 2k\left(\sqrt{R^2 - z^2} - r_0\right)^2 - m(R^2 - z^2)\omega^2$$

- b) (8 pts) Use the Euler-Lagrange equation to derive an equation of motion for the system.
- c) (14 pts) Use the effective potential  $V_{\text{eff}}(z)$  (calculated in a) to determine the equilibrium coordinates  $z_e$  for the beads. Show that there exists a critical angular velocity

$$\omega_c = \sqrt{\frac{2k}{m} \left(1 - \frac{r_0}{R}\right)} = \omega_0 \sqrt{1 - \frac{r_0}{R}}$$

where  $\omega_0^2 \equiv 2k/m$  such that there are three possible equilibria for  $\omega < \omega_c$  and only one possible equilibrium for  $\omega > \omega_c$ .

**Problem 2 (30pts)**

- a) (15 pts) Use the Poisson brackets to find the values of  $\alpha$  and  $\beta$  such that the following transformation is canonical:

$$Q = q^\alpha \cos(\beta p), \quad P = q^\alpha \sin(\beta p)$$

- b) (15 pts) A physical system is characterized by a time-independent Hamiltonian  $H_0(p, q)$  with  $(p, q)$  as the conjugate pair of canonical variables for the system. Now, the system is being perturbed by an external oscillating field so that the Hamiltonian becomes:

$$H = H_0(p, q) - \varepsilon q \sin(\omega t)$$

where  $\varepsilon$  and  $\omega$  are respectively the magnitude and angular frequency of the periodic perturbation.

- i. Write down the Hamilton equation of motion for this system.
- ii. Given the following generating function:

$$F_2(q, P, t) = qP - \frac{\varepsilon q}{\omega} \cos(\omega t)$$

calculate the resulting canonical transformation  $(q, p) \rightarrow (Q, P)$ .

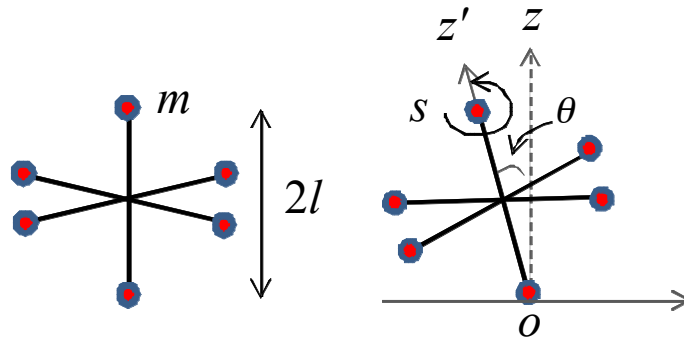
- iii. What is the new Hamiltonian  $K(Q, P)$  in the new canonical variables?

Show that the Hamilton equation of motion in these new canonical variables is in the standard form:

$$\dot{Q} = \frac{\partial K(P, Q)}{\partial P}$$
$$\dot{P} = -\frac{\partial K(P, Q)}{\partial Q}$$

**Problem 3 (30pts)**

“Jacks” is a childhood game involving metal pieces that can be thought of as six small equal masses  $m$  connected by a set of orthogonal massless rods of length  $2l$  as shown.



- (8 pts) Choosing the origin  $O$  of the body axis as the contact point between the jack and the ground, calculate the principal moments of inertia for the jack.
- (8 pts) Using the three Euler's angles, write out the components of the torque on the jack due to gravity in the body frame if it is tilted slightly off the vertical as shown above.
- (8 pts) If you spin the jack around the principal axis  $z'$  (see figure above) so that there is a steady precession around the vertical, what is the relation between the spin angular velocity  $s$ , the precession rate  $\Omega$ , and the angle  $\theta$  between the  $z$ -axis in the space frame and the  $z'$ -axis in the body frame?
- (6 pts) Assuming the jack is spinning near the vertical ( $\theta \ll 1$ ), show that the precession is stable if the spin rate  $s$  satisfies the following condition:

$$s > \frac{3\Omega}{2} + \frac{3g}{2l\Omega}$$