# Qualifying exam - August 2016

## Statistical Mechanics

You can use one textbook. Please write legibly and show all steps of your derivations. Note the Formula Sheet attached.

**Problem 1** [20 points]
Consider a system of non-interacting identical localized oscillators. Using the classical Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2, \tag{1}$$

($m$ is the particle mass and $x$ displacement from equilibrium) calculate

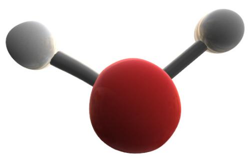
1. [10 points]
$$\overline{\left(x^2 - \overline{x^2}\right)^2}. \tag{2}$$
2. [10 points]
$$\overline{\left(p^2 - \overline{p^2}\right)^2}. \tag{3}$$

**Problem 2** [25 points]
An adiabatic rigid cylinder is divided in two compartments by a piston. One compartment is filled with $N_1 = 13$ moles of water vapor at a temperature $T_1$ and pressure $p_1$. The other compartment is filled with $N_2 = 12$ moles of carbon dioxide $CO_2$ at a temperature $T_2 = 3T_1$ and pressure $p_2 = 2p_1$. Each gas is initially in thermodynamic equilibrium. The piston is removed and the gases mix. After equilibrium has been reached,

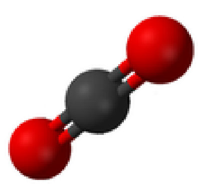
1. [5 points] What is the internal energy of the gas mixture in the cylinder?
2. [10 points] What is the temperature of the gas mixture?
3. [10 points] What is the pressure of the gas mixture?

Consider both gases as ideal and treat the molecular rotations and atomic vibrations using classical mechanics.

$H_2O$



$CO_2$



**Problem 3** [20 points]
Consider a cavity containing black-body radiation at a temperature $T_1$. Suppose the volume of the cavity increases in an equilibrium adiabatic process from an initial value $V_1$ to a final value $V_2 = 5V_1$.

1. [5 points] What is the final temperature $T_2$ in the cavity?
2. [5 points] If the initial radiation pressure was $p_1$, what is the final pressure $p_2$?
3. [10 points] If the cavity initially contained a total of $N_1$ photons, what is the final number $N_2$ of photons in the cavity? Explain the physical meaning of this result.

**Problem 4** [25 points]
Consider a free electron gas at $T = 0$ K. Suppose its volume is $V$ and the number of electrons is $N$.

1. [5 points] Show that the total kinetic energy of the gas is

$$U_0 = \frac{3}{5} N \varepsilon_F, \tag{4}$$

where $\varepsilon_F$ is the Fermi energy.

2. [5 points] Derive the following relation between the gas pressure $p$ and total energy $U_0$:

$$pV = \frac{2}{3} U_0. \tag{5}$$

3. [5 points] Show that the isothermal compressibility of the gas, $\beta_T = -\left(\partial \ln V / \partial p\right)_{T,N}$, equals

$$\beta_T = \frac{3V}{2N\varepsilon_F}. \tag{6}$$

4. [5 points] The speed of sound in a gas is given by

$$v_s = \left[\left(\partial p / \partial \rho\right)_T\right]^{1/2}, \tag{7}$$

where $\rho$ is the gas density (mass per unit volume). Compute $v_s$ for the free electron gas at $T = 0$ K and compare it with the Fermi velocity $v_F$.

5. [5 points] If $v$ is the electron speed, calculate $\overline{v}$, $\overline{(1/v)}$, and show that $\overline{v}\,\overline{(1/v)} > 1$.

## Formula Sheet

Moments of the Gaussian function:

$$M_n = \int_0^\infty x^n e^{-x^2} dx. \tag{8}$$

Selected values: $M_0 = \sqrt{\pi}/2$, $M_1 = 1/2$, $M_2 = \sqrt{\pi}/4$, $M_3 = 1/2$, $M_4 = 3\sqrt{\pi}/8$, $M_5 = 1$, $M_6 = 15\sqrt{\pi}/16$.