Qualifying exam-January 2021

Statistical Mechanics

You can use one textbook. Please write legibly and show all steps of your derivations.

Problem 1 [20 points].

1) Based on Bekenstein and Hawking theory, the entropy of a black hole is proportional to its area A, and is given by:

$$S = \frac{k_B c^3}{4G\hbar} A ,$$

where the relation between the radius and mass of a black hole is given by

$$R = \frac{2G}{c^2} M ,$$

a) (10 points) How does the entropy change when two black holes collapse into one?

b) (10 point) The internal energy of the black hole is given by Einstein relation, $E = Mc^2$. Find the temperature of the black hole in terms of its mass.

Problem 2 [25 points].

Consider a two-dimensional solid material composed of N atoms. Assume that the atomic vibrations are harmonic and have 2 polarizations: one longitudinal and one transverse, with the speeds of sound $u_L$and u_T, respectively.

(a) (10 points) Derive the Debye density of states and frequency of the material.

(b) (15 points) Compute the heat capacity in the high temperature and low temperature limits. You may start with the relation for heat capacity associated with one individual vibrational mode.

Problem 3 [35 points]

Consider a gas of N non-interacting identical non-relativistic fermions of mass m trapped in a 1D harmonic potential where the energy spectrum is given by:

$$\epsilon_n = \hbar\omega(n + \frac{1}{2})$$

The gas is in equilibrium at temperature T. For simplicity, ignore the spin degeneracy of each level.

(a) (4 points) Find the Fermi energy ε_F of the gas at T = 0 as a function of N.

(b) (8 points) Calculate the exact total energy per particle E/N at T = 0.

(c) (5 points) Calculate the grand canonical partition function $\Gamma(\mu, N)$ where μ is the chemical potential.

(d) (5 points) Calculate the grand potential. Do not attempt to evaluate the infinite sum.

(e) (5 points) Derive the average total number of particles $\overline{N}(\mu, T)$. Do not attempt to evaluate the infinite sum.

(f) (8 points) Find an explicit expression for $\overline{N}(\mu, T)$ in high temperature regime ($k_B T \ll \hbar\omega$) and $\zeta \ll 1$,

where $\zeta = e^{\beta\mu}$ is the fugacity of the gas.

You might use the following formulas for this problem:

$$\sum_{n=0}^{N-1} n = \frac{N(N-1)}{2}$$

$$\sum_{0}^{\infty} x^n = \frac{1}{1-x}$$

Problem 4 [20 points].

Consider a Bose gas that follows a linear energy-momentum relation $\varepsilon = v|p|$ in a space of dimensionality d = 1 and d = 2. Assume the Bose gas has spin 1.

a) (10 points) In which of these dimensions will the Bose-Einstein condensation occur?

b) (10 points) For the case where the Bose-Einstein condensation occurs, find the Bose-Einstein transition temperature T_C.

Mathematical Formulas:

$$I_n = \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx = (n-1)!\,\zeta(s) \quad , where\ \zeta(s)\ is\ Riemann\ zeta\ function$$

s	1	2	3	4
$\zeta(s)$	∞	$\frac{\pi^2}{6}$	1.202	$\frac{\pi^4}{90}$