Qualifying exam - January 2020

Statistical Mechanics

You can use one textbook. Please write legibly and show all steps of your derivations. Note the Formula Sheet attached.

Problem 1 [26 points]

Consider a system of non-interacting identical localized oscillators. Using the classical Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2, \tag{1}$$

(m is the particle mass and x displacement from equilibrium) calculate

1. [10 points]
$$\overline{\left(x^2 - \overline{x^2}\right)^2}. \tag{2}$$

2. [16 points]
$$\overline{\left(p^2 - \overline{p^2}\right)^2}. \tag{3}$$

Problem 2 [24 points]

Consider a free electron gas at $T = 0$ K. Suppose its volume is V and the number of electrons is N.

1. [4 points] Show that the total kinetic energy of the gas is

$$U_0 = \frac{3}{5}N\varepsilon_F, \tag{4}$$

where ε_F is the Fermi energy.

2. [5 points] Derive the following relation between the gas pressure p and total energy U_0:

$$pV = \frac{2}{3}U_0. \tag{5}$$

3. [5 points] Show that the isothermal compressibility of the gas, $\beta_T = -\left(\partial \ln V / \partial p\right)_{T,N}$, equals

$$\beta_T = \frac{3V}{2N\varepsilon_F}. \tag{6}$$

4. [5 points] The speed of sound in a gas is given by

$$v_s = [(\partial p / \partial \rho)_T]^{1/2}, \tag{7}$$

where ρ is the gas density (mass per unit volume). Compute v_s for the free electron gas at $T = 0$ K and compare it with the Fermi velocity v_F.

5. [5 points] If v is the electron speed, calculate $\bar{v}$, $\overline{(1/v)}$, and check if the inequality $\bar{v}\overline{(1/v)} > 1$ is satisfied.

Problem 3 [24 points]

Imagine a harmonic solid with an isotropic dispersion relation $\omega = Ak^b$, where ω is the angular frequency of atomic vibrations, k is the wave number, and $A > 0$ and $b > 0$ are constants. Assuming that this dispersion relation holds for each of three polarizations of phonons, show that in the low-temperature limit the phonon contribution to the heat capacity of the solid is proportional to $T^{3/b}$.

Problem 4 [26 points]

A system has two quantum states, state 0 with energy 0 and state 1 with energy ε. These states can be occupied by non-interacting fermions from a particle and heat reservoir at a temperature T and chemical potential μ.

1. [6 points] Calculate the grand partition function $\Gamma(T, \mu)$ of the system.

2. Using the obtained $\Gamma(T, \mu)$, compute the following properties as functions of T and μ:

   (a) [6 points] Average occupation numbers of the two states, $\bar{n}_0$ and $\bar{n}_1$.
   (b) [6 points] Average total energy $\bar{E}$.
   (c) [8 points] The system entropy S.

Formula Sheet

Moments of the Gaussian function:

$$M_n = \int_0^\infty x^n e^{-x^2} dx. \tag{8}$$

Selected values: $M_0 = \sqrt{\pi}/2$, $M_1 = 1/2$, $M_2 = \sqrt{\pi}/4$, $M_3 = 1/2$, $M_4 = 3\sqrt{\pi}/8$, $M_5 = 1$, $M_6 = 15\sqrt{\pi}/16$.