Qualifying exam - Fall 2021

Statistical Mechanics

You can use one textbook. Please write legibly and show all steps of your derivations.

Problem 1 [20 points]**:**

Consider a one-dimensional quantum harmonic oscillator in thermal equilibrium with a heat reservoir at temperature T, where the energy spectrum is given by:

$$\epsilon_n = \hbar\omega(n + \frac{1}{2})$$

(a) (10 points) Calculate the mean value of the oscillator's energy as a function of T.
(b) (5 points) Calculate the root-mean-square fluctuation (variance) in energy.
(c) (5 points) Find the quantity calculated in part (a) in the limits $k_B T \ll \hbar\omega$ and $k_B T \gg \hbar\omega$.

Problem 2 [30 points]**:**

Consider a substance with the fundamental equation

$$U = Aexp(\alpha S + \beta M^2)$$

where U is energy, M is the total magnetic moment, S is the entropy and A, α and β are some constants which can depend on N.

(a) [5 points] Using the 1st law of thermodynamics drive a relation for M as a function of T and H. (H is magnetic field inside the substance).
(b) [5 points] Calculate the isothermal magnetic susceptibility χ_T (T, H).
(c) [10 points] Calculate the adiabatic magnetic susceptibility χ_S (T, H). Find the small field limit H→0.
(d) [10 points] Derive a relation for Helmholtz Free energy F (T, M).

Problem 3 [20 points]:

Consider a system of two identical non-interacting particles in a canonical ensemble at temperature T. Each particle can be in 5 different states with energies $E = n\varepsilon$, where n=0,1, 2, 3, 4.

(a) [5 points] Compute the partition function of the system in a classical Maxwell-Boltzmann approximation.

(b) [5 points] Compute the partition function of the system assuming Fermi-Dirac statistics.

(c) [5 points] Compute the partition function of the system assuming Bose-Einstein statistics.

(d) [5 points] Compare these partition functions in the limit of high temperatures ($k_B T \gg \varepsilon$) and low temperatures ($k_B T \ll \varepsilon$).

Problem 4 [30 points]:

Consider a Bose gas that follows a linear energy-momentum relation $\varepsilon = v|p|$ in a space of dimensionality d = 1 and d = 2. Assume the Bose gas has spin 1.

a) (20 points) In which of these dimensions will the Bose-Einstein condensation occur?

b) (10 points) For the case where the Bose-Einstein condensation occurs, find the Bose-Einstein transition temperature T_C.

Mathematical Formulas:

$$I_s = \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx = (s-1)!\,\zeta(s) \quad , where\ \zeta(s)\ is\ Riemann\ zeta\ function$$

s	1	2	3	4
$\zeta(s)$	∞	$\frac{\pi^2}{6}$	1.202	$\frac{\pi^4}{90}$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$