Quantum Mechanics Qualifying Exam

Spring 2021

January 20 (9:00 am - 12:00 pm)

1. Identical non-interacting particles of mass m are loaded into a *two-dimensional* square box of side length L with a flat potential ($V = 0$) and impenetrable walls. Determine the lowest four energy levels and their degeneracies for *one particle* inside the box. Then, using this, find the lowest-energy states of *a group of particles*, and determine their total energies and degeneracies in the following cases:

   (a) Three lowest-energy levels of three spinless bosons.

   (b) Two lowest-energy levels of three spin $\frac{1}{2}$ fermions.

2. A particle of mass m moves in one dimension under the influence of some potential $V(x)$. Its wavefunction is given by:

$$\psi(x,t) = Ae^{-\alpha x^2 - \beta x}e^{-i\omega t}$$

   (a) Find the potential $V(x)$.

   (b) Find the mean position and mean momentum of the particle.

   (c) Find the probability $P(p)dp$ that the particle's momentum is between p and $p + dp$.

3. (a) The normalized p-wave spherical harmonics expressed in spherical coordinates are:

$$Y_1^1(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi} \quad , \quad Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta \quad , \quad Y_1^{-1}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi}$$

   Rewrite them in Cartesian coordinates using:

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

   Show that:

$$Y_1^{+1} - Y_1^{-1} = \sqrt{\frac{3}{2\pi}}\,\frac{x}{r} \quad , \quad Y_1^{+1} + Y_1^{-1} = i\sqrt{\frac{3}{2\pi}}\,\frac{y}{r}$$

   (b) Consider a particle with the wavefunction:

$$\psi(x,y,z) = N(x - 2y + 2z)e^{-\alpha r}$$

   where $r = \sqrt{x^2+y^2+z^2}$, N is a normalization constant and α is a parameter. Use the formulas from the part (a) to calculate the probabilities $P(m)$, $m \in \mathbb{Z}$ that a measurement of the angular momentum L_z about the z-axis in the state $\psi(x,y,z)$ would yield $m\hbar$.

4. A collection of electrons is prepared by measuring their S_y spin projection and selecting only those that yield $s_y = +\frac{\hbar}{2}$. Then, a magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ is turned on at $t = 0$. The Hamiltonian is $H = -\mu\mathbf{SB}$.

   (a) Calculate the amplitudes $C_\uparrow(t)$ and $C_\downarrow(t)$ in the electrons' state $|\psi(t)\rangle = C_\uparrow(t)|\uparrow\rangle + C_\downarrow(t)|\downarrow\rangle$ at time $t > 0$, where $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of S_z corresponding to the eigenvalues $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ respectively.

   (b) What is the probability that the electron would be detected with its spin pointing in the positive x direction at time $t > 0$?

Useful formulas:

- Gaussian integral:

$$\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$$

- Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$e^{i\boldsymbol{\sigma}\hat{\mathbf{n}}\theta} = \cos\theta + i(\boldsymbol{\sigma}\hat{\mathbf{n}})\sin\theta$$