Physics Qualifying Exam, August 2020

Quantum Mechanics

Note: This is an open book exam and you are recommended to use the "Modern Quantum Mechanics" by J. J. Sakurai and Jim J. Napolitano. If a formula appears in the book, please use that as a starting point, there is no need to show the derivation of that formula.

1. [16 pts]

   (a) Explain why the stationary states of a one-dimensional harmonic oscillator have real wavefunctions, while some stationary states of a hydrogen atom have complex wavefunctions.

   (b) Write the general three-dimensional eigenfunction for a free particle of mass m and momentum $\mathbf{p}$ that represents the simultaneous eigenstate of energy and momentum.

   (c) Using a symmetry argument, explain why the commutator $[x, p] = i\hbar$, has i on the right-hand side.

   (d) Write the general angular momentum part of the wavefunction for a particle moving in a central force (aligned radially and dependent only on the distance from the origin). Can such a wavefunction describe the motion of a free particle? Explain your answer.

2. [16 pts] Consider a system of three non-interacting particles of mass m confined in a three-dimensional potential $V(\mathbf{r}) = V_0(x^2 + y^2 + z^2)$. Calculate the total ground state energy E_0 and the first excited state energy E_1 of the system in the following cases:

   (a) All particles are electrons.

   (b) All particles are helium-4 atoms.

   (c) Two particles are electrons and one particle is a positron.

3. [20 pts] At $t = t_0$, a particle of mass m in a one-dimensional world is equally likely to be in the ground and the first excited state of the potential:

$$V(x) = \begin{cases} 4kx^2 & , \quad x > 0 \\ \infty & , \quad x < 0 \end{cases}$$

   (a) What is the wave function of the particle at $t = 2t_0$?

   (b) What are the expectation values of its energy and momentum at $t = t_0$ and at $t = 2t_0$?

4. [16 pts]

(a) Show that the expectation value of momentum $\langle p \rangle$ is zero if the wave function $\psi(x)$ is real-valued.

(b) Let a particle have the mean momentum $\langle p \rangle$ in the state described by the wavefunction $\psi(x)$. What is its mean momentum in the state described by the wavefunction $e^{-ibx/\hbar}\psi(x)$?

5. [16 pts] A free particle of mass m in a one-dimensional world is described by a Gaussian wave function of width $\sqrt{\langle (x - \langle x \rangle)^2 \rangle} = \sigma$. If the average position of the particle is $\langle x \rangle = 0.5\sigma$ and average momentum is $\langle p \rangle = p_0$, what is the wave function of the particle? You need not worry about the normalization. Is this wavefunction an eigenfunction of position, momentum, or energy? Under what condition does this wavefunction describe an eigenfunction of a harmonic oscillator?

6. [16 pts] Construct the unitary operator U that performs each of the followinf operations:

(a) Spatial translation by δx: $U|x\rangle = |x + \delta x\rangle$

(b) Momentum boost by δp: $U|p\rangle = |p + \delta p\rangle$

(c) Time evolution by δt: $U|\psi(t)\rangle = |\psi(t + \delta t)\rangle$

(d) Rotation about the z-axis by $\delta\phi$: $U|\theta, \phi\rangle = |\theta, \phi + \delta\phi\rangle$

(e) Spin rotation from x to y direction: $U|S_x, +\rangle = |S_y, +\rangle$

(f) Time reversal: $U\psi(x,t) = \psi(x, -t)$