Classical Mechanics Qualifier (January 2022)
George Mason University

You are allowed to use your graduate textbook during the exam.
Four problems | Total 100 points

Problem 1 (20pts)

A system is described by the Hamiltonian, $H = \frac{p^2}{2} - \frac{1}{2q^2}$. Write down the Hamilton's equation of motion for this system. Show that $F = \frac{pq}{2} - Ht$ is a constant of motion for this system.

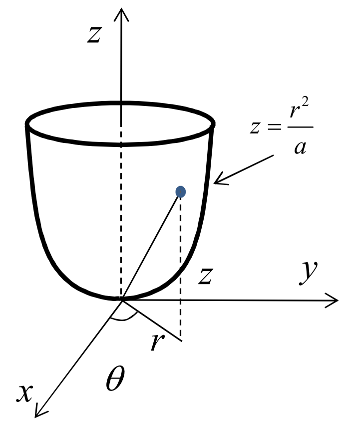
Problem 2 (20pts)

Consider the orbits of a mass *m* in a central inverse-cube force, $F = \frac{-k}{r^3}$, where *k* is a positive constant. Solve the radial equation of motion for *r* as a function of the angle variable θ on the orbital plane and classify them as bound or unbound for each of the following three cases:

i. Large angular momentum: $l > \sqrt{mk}$
ii. Small angular momentum: $l < \sqrt{mk}$
iii. $l = \sqrt{mk}$

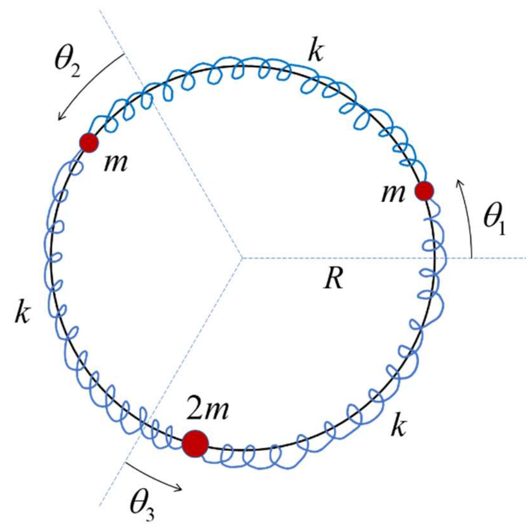
(l is the constant generalized momentum corresponding to generalized coordinate θ.)

Problem 3 (30pts)



A particle of mass m is constrained to move under the influence of gravity on the inside of a smooth parabolic surface of revolution given by $r^2 = az$. Use the Lagrange undetermined multiplier method to derive the constraint force for this problem. Write your answer as a vector in cylindrical coordinates. (Hint: You might want to use the two constants of motion E and l to simplify some of your expressions. The magnitude of the constraint force is proportional to $\left(1+\frac{4r^2}{a^2}\right)^{-3/2}$.)

Problem 4 (30pts)



Three beads with mass, *m, m,* and 2*m* slide frictionlessly on a thin hoop of radius *R*. They are connected by springs that wrap around the hoop. The springs all have the same force constant *k*. (The springs also slide frictionlessly around the hoop.) When the beads are equally spaced around the hoop, the springs are unstretched. There is no gravity.

a) Write the Lagrangian of the system using the angles θ_1, θ_2, and θ_3, which are measured from equally spaced, but arbitrary, positions around the ring 120° apart.
b) Find the normal frequencies of the system.
c) Find the corresponding normal modes.
d) Describe qualitatively the relative motion of the three masses for each of the normal modes found in (c).