Classical Mechanics Qualifier Fall 2018 George Mason University

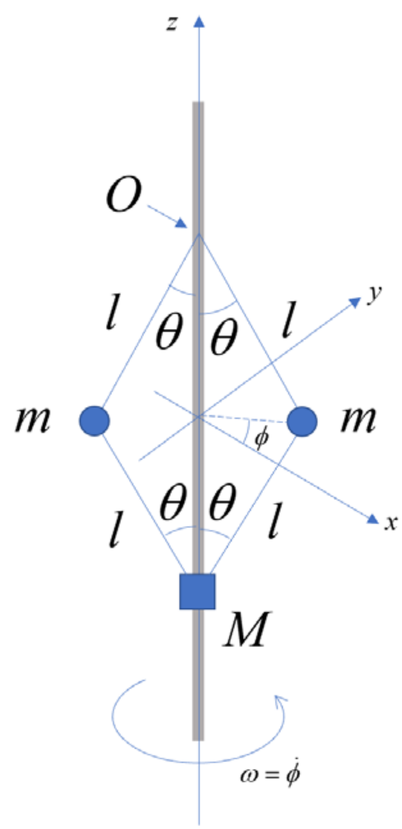
There are four questions, and each is worth 20 points.

Please write your solutions on the blank paper that is provided for you.

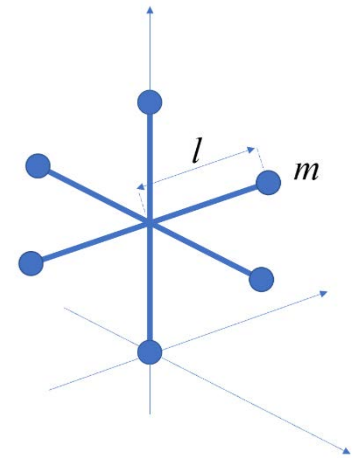
If anything is ambiguous, say so on your paper and explain why. If you need to make any assumptions in order to solve a problem, state your assumptions explicitly.

1. The diagram shows a flyball governor for a steam engine. This consists of two masses m connected to arms of length l and another mass M as shown. The arms are attached to a vertical central shaft with hinges, and mass M can slide up and down on the shaft. (θ is restricted to be between 0 and 90 degrees.) The connection point at O does not move up and down. The whole assembly is constrained to rotate with constant angular velocity $\omega = \dot{\phi}$ around the shaft, such that the two masses m move in circular paths in a plane that is perpendicular to the z-axis. Neglect all friction and the mass of the arms, and treat all masses as point masses.

a) Using the Cartesian coordinate system shown, obtain the Lagrangian of this system. Express it in terms of constants and the single generalized coordinate θ.
b) Obtain the equation of motion.
c) Discuss the conditions for the existence of equilibria for θ.



2. Jacks is a childhood game involving metal pieces that can be thought of as six small equal point masses m connected by a set of orthogonal massless rods of length $2l$ as shown.



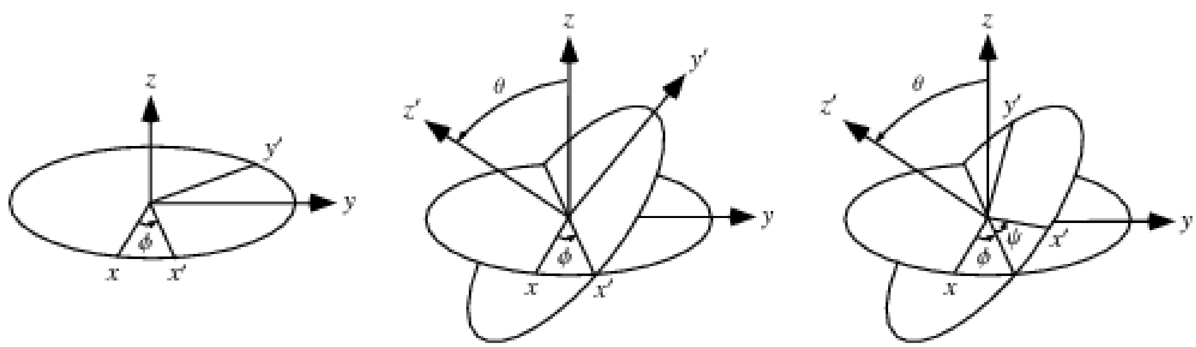
a) As the diagram suggests, choose body axes that are parallel to the jack's rods, and place the origin O at the contact point between the jack and the ground. Using this coordinate system, calculate the principal moments of inertia for the jack.

b) Let the position of the jack relative to a fixed coordinate system that shares the same origin (the "space frame") be specified by the Euler angles ϕ, θ, ψ (see diagram below). The net force on the jack due to gravity acts on the center of mass of the jack, and expressed relative to the space frame, is $(0, 0, -6mg)$. Obtain the components of both the force and the torque on the jack due to gravity relative to the <u>body</u> frame.

c) Now assume that the jack is spun very quickly about its body z'-axis (i.e., the body axis that contains the origin and two of the masses) with angular speed s and is released. Express the components of the angular velocity vector relative to the body coordinates in terms of the Euler angles.

d) Using the Euler equations of motion, find the relationship between s, the rate of precession about the space z-axis $\Omega = \dot{\phi}$, and θ.

e) Obtain an approximation for the rate of precession Ω assuming s is large. (You may or may not want to use Euler's equations for this.)



3. Suppose a point particle moves in space subject central force given by

$$F(r) = -\frac{k}{r^2} - \frac{3a\ell^2}{r^4}$$

where ℓ is the angular momentum (assumed to be non-zero) of the particle and $r = 0$ is the origin.

a) Find the potential energy function $U(r)$ corresponding to this force. Let $U \to 0$ as $r \to \infty$.
b) Obtain the effective potential energy function and show that the existence of circular orbits is mediated by the relationship between ℓ^2 and $12ka$.
c) Sketch the effective potential for the relevant cases and describe qualitatively the possible orbits with respect to the system's total energy E.

4. Derive the formulas that yield the canonical transformation $Q = Q(q, p), P = P(q, p)$ that is generated by

$$F(q, p, Q, P) = -\left[\mathrm{e}^{Q} - 1\right]^2 \tan p + qp.$$

Use those formulas to obtain explicit expressions for $Q(q, p)$ and $P(q, p)$.